On the Possibility of Informative Equilibria in Contingent Markets with Feedback*

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Abstract

We consider the operation of contingent markets in the presence of an intervening agent who is able to alter probability of the event that the market trades on in response to the market price. Such description presumably characterizes, for instance, prediction markets created with the intention of forecasting adverse future events such as a financial crisis or a terrorist attack. We add an intervening agent in a model of a market for a contingent claim, and show that this addition diminishes substantially the information aggregation capabilities of the market. In particular, we show that: (*i*) there are reasonable circumstances in which a (fully or partially revealing) rational expectations equilibrium does not exist; (*ii*) the intervention may make the market price to *decline* in response to information ex-ante *more favorable* to the occurrence of the underlying event; (*iii*) in some cases there are equilibria that reveal no information. These results imply that in the presence of intervention the market may cease to function as an information aggregator.

Keywords: prediction market, rational expectations equilibrium, information aggregation

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1 Introduction

Contingent markets are futures markets where traders can "bet" on the occurrence of a future event by buying or selling contracts whose payoffs are contingent on the occurrence of the event of interest. An example of contingent markets are "prediction markets" (also known as "information markets"): they trade on some type of Arrow-Debreu contingent claim, where the issuer (seller) promises to pay the holder of the claim $1 if a specific event occurs and zero otherwise. Perhaps the best known prediction markets are the Iowa Electronic Markets on U.S. election outcomes, run by the Tippie College of Business at the University of Iowa.[1] In their review article Wolfers and Zitzewitz (2004) provide many other examples, along with details on the institutional setup of these markets.

It has been documented by several authors that contingent markets are capable of outperforming polls and other traditional methods in forecasting future events; see, for example, Forsythe, Nelson, Neumann, and Wright (1992), Plott (2000), Berg, Forsythe, Nelson, and Rietz (2003) and Berg, Nelson, and Rietz (2008).[2] Presumably, the success of these markets in forecasting the future derives from their ability to aggregate beliefs and information that individual traders possess about the likelihood of the underlying event. For instance, Manski (2006) and Wolfers and Zitzewitz (2007) find that the equilibrium price of an Arrow-Debreu security written on an event is given by the mean or a certain quantile of the distribution of beliefs among traders about the probability of that event. (Differences in their results are due to different assumptions on traders' attitudes toward risk.) As information is not explicitly modeled in either paper, the way in which the market price aggregates information is unclear. Ottaviani and Sørensen (2015) obtain deeper aggregation results by explicitly introducing an information structure.

Given the success of contingent markets in forecasting future events, it seems natural to take the next step and "[use] the information in market values to make decisions" (Berg and Rietz (2003)). Along the same lines, Hanson (1999) proposes "decision markets" cre-

[1] http://www.biz.uiowa.edu/iem/

[2] Jacobsen, Potters, Schram, van Winden, and Wit (2000), on the other hand, finds that an European electoral prediction market which run through several elections was less accurate predicting election results than most election markets based on U.S. elections.

ated specifically for the purpose of evaluating various policy alternatives. Nevertheless, the existing literature on contingent markets does not explore the consequences of the idea that forecasts produced from prices might be used in subsequent decisions. The key point we want to make in this paper is that these decisions could have an impact on the probability of occurrence of the very event the market was meant to forecast. We will refer to this as the "feedback problem." In particular, all previously mentioned aggregation results assume implicitly that no trader, nor any other agent, can affect the probability of the underlying event by taking any action. Yet, it is easy and natural to envision scenarios where this is indeed the case. To illustrate this situation, we present the following example.

Example 1: Consider a prediction market where Arrow-Debreu claims contingent on the occurrence of a financial crisis in country X are traded. The government (or central bank) of country X considers the market price as an indicator of the probability of a crisis. Hence, if the price of the claim starts rising, the government may choose to take corrective action in the economy. Such action will affect the probability of the crisis occurring, and it is also observable by traders in the market. As a consequence, the intervention will have an effect on the market price. In fact, rational traders are expected to anticipate the government's reaction and incorporate these expectations into their trading behavior ahead of time. ∎

A second (real world) example is DARPA's infamous 2003 proposal of a market in "terrorism futures."[3] As documented by Hanson (2006), who was closely involved with the project, the public reception of the idea was extremely unfavorable, and the project was quickly abandoned primarily out of moral concerns.[4] Hanson (2006) acknowledges the problem of potentially "self-defeating prophecies" in these markets, and proposes contracts *conditional* on the level of response as a solution. Nevertheless, it is still important to understand how the possibility of intervention affects trading of a contract that does not explicitly condition on any response. (To our knowledge, contracts traded on all existing prediction markets are

[3]DARPA stands for "Defense Advanced Research Project Agency" and is affiliated with the U.S. Department of Defense.

[4]Meirowitz and Tucker (2004) discusses DARPA's defunct proposal in detail, and proposes a number of design features that they claim would make this type of information market work well. However, they do not address the feedback problem, which is the main focus of our paper.

not conditioned on any intervention.) Presumably, in this case the market will implicitly condition on what it sees as the optimal level of response given the information contained in the market price, and this implicit conditioning might actually undermine the viability of the market as an early warning device.

As the examples illustrate, there are plausible situations in which some agent has the ability and intention to alter the probability of the event that is being traded on, potentially in response to the market price. The first contribution of the paper is to draw attention to the possibility of such feedback in certain contingent markets in particular, and in futures markets in general. Second, we formally analyze how feedback affects the existence of informational equilibria and the relationship between the observed price and traders' information. The introduction of possible feedback is a substantive addition to the usual theoretical setup in the existing literature, and leads to new and interesting results. In particular, we find that a fully revealing rational expectations equilibrium (REE) may exist in some special circumstances, but the existence of such equilibrium is less likely than when intervention is absent; furthermore, we show that even when a fully revealing REE exists, the equilibrium price may *rise or fall* in response to information that in itself is favorable to the occurrence of the event in question, a result that again is in sharp contrast with previous results derived from the no-intervention case. Next, state conditions that preclude the existence of a (fully or partially revealing) REE. Lastly, we show that non-revealing equilibria exist in some circumstances. In the last two cases the contingent market cannot fulfill its role as an information aggregator.

The broader idea behind our paper—that forecasts might affect the realization of the target variable—has been raised previously in a number of other contexts. In an early paper, Grunberg and Modigliani (1954) analyze the problem of a forecaster who wants to predict the price of a commodity at a future date, but whose forecast affects supply, and hence the price, in that period. Grunberg and Modigliani (1954) show that under general conditions there is a correct price forecast, in the sense that if that price is predicted then it will actually occur.

More recently, Ottaviani and Sørensen (2006) consider a feedback problem in forecasting games where professional forecasters try to predict the future price of an asset with a given "fundamental value." Forecasts affect the price as they reveal forecasters' private information to the market, but they do not affect the exogenously given fundamental value. On the other hand, in our case contingent market prices (i.e. forecasts) do affect the fundamental value of the contingent claim, since they alter the probability of the underlying event through price-based intervention. In another study, Ottaviani and Sørensen (2007) consider the problem of outcome manipulation in prediction markets, where traders have the ability to affect the probability of the underlying event, and they do so in an effort to profit from their trading positions. This problem is also different from ours. In our framework it is the information revealed by the market price that triggers intervention, and the intervening agent is not a trader in the market.

The rest of the paper is organized as follows. In Section 2 we introduce a simple model of a market in Arrow-Debreu securities that allows for feedback (but also accommodates no-feedback as a special case). In Section 3 we study the consequences of intervention in the information aggregation properties of the market. Despite its simplicity, the proposed model captures the main features of the feedback problem well, and produces a number of interesting results outlined above. In Section 4 we summarize our findings and conclude.

2 The basic model

A population of risk neutral traders[5] buy and sell Arrow-Debreu contingent securities written on the realization of a Bernoulli random variable z. The contingent claims pay \$1 if $z = 1$ and \$0 otherwise. In addition to traders, there is an agent denoted G (the "government") who does not trade in the market, and for whom the occurrence of $z = 1$ is harmful. Agent G has at its disposal a maximum amount of resources $A > 0$ to be potentially employed to reduce the likelihood of $z = 1$. Agent G's problem is to decide on the appropriate level of preventive action $a \in [0, A]$ so as to minimize cost of prevention plus expected damage from

[5] The assumption of risk neutrality is common in the literature on prediction markets, and a justification for it is that the most popular prediction markets have rather small investment limits. For instance, the Iowa Electronic Markets impose a limit of \$500 per account due to strict no-gambling laws in the U.S.

the possible occurrence of $z = 1$. The damage caused to G when $z = 1$ is assumed to be a fixed amount K.

The set of possible states of the world is $\Omega \times \Delta$, with $\Omega = \{0, 1\}$, $\Delta = [0, D]$, and generic element (ω, δ). The random variable z on which the market trades is defined as

$$z(\omega, a) = \omega 1_{\{\delta > a\}} \tag{1}$$

where $1_{\{.\}}$ is the indicator function. This setup models the effect of intervention in the following way: (i) For $\omega = 0$, it is common knowledge that $z = 0$; (ii) the conditional probability of $z = 1$ given $\omega = 1$ can be influenced by agent G through the intervention variable a, and it is equal to $Prob(\{\delta > a|\omega = 1\})$.[6] The following examples help in the interpretation of the proposed framework:

Example 2: $z = 1$ is the occurrence of a flood in a given town. For a flood to occur, the precipitation level has to be higher than a given limit. Therefore, $\omega = 0$ means that the amount of rain is lower than the minimum flood level, and $\omega = 1$ means that it is higher. However, even with enough rain, local authorities (G) could potentially prevent a flood by building a dam. In this case a can be interpreted as the height of the dam, and δ as the amount of rain above the minimum flood level. ∎

Example 3: Terrorists plan an attack on a given target ($\omega = 1$) or not ($\omega = 0$). If an attack is planned, then a successful attack ($z = 1$) will depend on the effectiveness of intelligence and security employed at the target (a), and on the amount of effort and skill level of the terrorists (δ). ∎

Traders are represented by the elements t of an index set T endowed with a measure μ. They hold heterogeneous prior beliefs q_t about the marginal probability of $\omega = 1$, and common prior beliefs about the distribution of δ conditional on $\omega = 1$, given by the density

[6] Formally, the event in which the contingent claim pays \$1 is

$$\{z = 1\} = \{1\} \cap [a, D]$$

function $r(\delta)$ which is strictly positive in the support $[0, D]$.[7] Traders have trading budgets y_t. We assume that y_t and q_t are distributed independently in the population of traders under the measure μ. The distribution of marginal prior beliefs q_t is given by the c.d.f. $F(.)$, where $F(.)$ is assumed to be continuous and strictly increasing on $[0, 1]$ with $F(0) = 0$ and $F(1) = 1$.[8] Trading budgets are distributed in the population according to the c.d.f. $B(.)$.

Given action $a \in [0, A]$ taken by agent G, let $\tilde{g}_t(a)$ denote trader t's assessment of the probability of $z = 1$ conditional on ω_1. Since traders hold common priors on δ conditional on ω, from the definition of z we have

$$\tilde{g}_t(a) = \tilde{g}(a) \equiv \int_a^D r(\delta) d\delta \tag{2}$$

The function $\tilde{g}(a)$ measures traders' perception about the effectiveness of intervention $a \in [0, A]$. We assume that G holds the same prior beliefs as traders about δ, and therefore $\tilde{g}(a)$ also describes G's perception about the usefulness of its intervention. It is easy to see from the definition that $\tilde{g}(a)$ is decreasing in a for $a \in [0, D]$. Hence, G's intervention in our model has the effect of reducing the probability of $z = 1$ given $\omega = 1$. Notice that our model nests the case in which no intervention is possible. In that case $A = 0$ and $\tilde{g}(0) = 1$. From the above assumptions it follows that if trader t believes, *a priori* or *a posteriori*, that the probability of $\omega = 1$ is b_t and action a is taken by G, then their assessment of the probability of $z = 1$ is given by $b_t \tilde{g}(a)$. Similarly, if G holds the belief that the probability of $\omega = 1$ is b_G and takes action a, then its assessment of the probability of $z = 1$ is $b_G \tilde{g}(a)$.

Traders receive a public signal $s(\omega)$ that is informative about ω (but not about δ conditional on ω) in that the conditional distribution of s given $\omega = 1$ differs from the conditional

[7] The distribution of δ conditional on $\omega = 0$ is degenerate, with $Prob[\delta = 0|\omega = 0] = 1$. For instance, if a terrorist attack is not planned to be attempted, the level of effort put into that attack is assumed to be zero with probability one, and therefore the probability of a successful attach is zero. Also, the assumption that traders hold common prior beliefs on δ conditional on $\omega = 1$ implies that they agree on the effectiveness of intervention. For instance, this assumption would mean that traders may disagree on the likelihood of a terrorist attack being planned, but they all concord on how likely it is that an attack will succeed given that it has been planned *and* intervention level takes on a given value.

[8] These assumptions imply that the index set T cannot be finite or even countable and μ must be non-atomic.

distribution of s given $\omega = 0$. The conditional density of the signal given ω is given by the likelihood function $h(s \mid \omega)$, and it is the same for all traders.[9] Trader t's posterior probability of $\omega = 1$ conditional on the signal s is computed by updating t's prior beliefs using Bayes' rule.[10] Trader t's posterior probability of $\omega = 1$ conditional on signal s, denoted by $\tilde{\pi}_t$, is given by

$$\tilde{\pi}_t = \frac{h(s \mid \omega = 1)q_t}{h(s \mid \omega = 1)q_t + h(s \mid \omega = 0)(1 - q_t)} = \frac{q_t L(s)}{1 - q_t[1 - L(s)]} \tag{3}$$

where

$$L(s) = \frac{h(s \mid \omega = 1)}{h(s \mid \omega = 0)}$$

Notice that the likelihood ratio $L(s)$ includes all the information about ω contained in the signal s as far as traders' decisions are concerned, i.e. $L(s)$ is a sufficient statistic for ω. In what follows we will write L for $L(s)$ to save on notation. Rearranging (3), we can write trader t's posterior as

$$\tilde{\pi}_t(L) = \frac{L}{\frac{1}{q_t} - 1 + L} \tag{4}$$

Action by G affects the probability of $z = 1$ given $\omega = 1$. We denote by π_t trader t's posterior probability of $z = 1$ conditional on L and a. Therefore

$$\pi_t(L, a) = \tilde{\pi}_t(L)\tilde{g}(a) \tag{5}$$

In our model G does not receive a private signal (or G's "prior" already includes any private information), so the only way its beliefs can be updated is by using information revealed by the market price.[11] Therefore, if the market reveals the likelihood ratio L, G's posterior probability of $\omega = 1$ is given by

$$\tilde{\pi}_G(L) = \frac{L}{\frac{1}{q_G} - 1 + L}, \tag{6}$$

[9] In the language of Milgrom and Stokey (1982), beliefs are "concordant."

[10] Bondarenko and Bossaerts (2000) provide evidence that traders in the Iowa Electronic Markets are rational Bayesian decision makers.

[11] In some applications one might plausibly assume that G also receives information correlated with ω, and to study also if the market price reveals such information. In the current paper we abstract from this possibility since our focus is on how G can take advantage of traders' information revealed by the market price and not viceversa.

where q_G denotes G's prior beliefs on the probability of $\omega = 1$. On the other hand, when the market does not reveal any information about the signal profile, agent G's posterior belief about ω is given by its prior belief q_G. Similarly to traders, when L is revealed by the price, G's posterior beliefs about the probability of $z = 1$ are

$$\pi_G(L, a) = \tilde{\pi}_G(L)\tilde{g}(a). \tag{7}$$

Intervention is costly for G. The direct cost of taking action a is given by the continuous function $c(a)$, which is assumed to be increasing. Agent G's objective is to minimize expected total cost, this is, the cost of intervention plus the expected damage from $z = 1$. When L is revealed by the price, G's optimal action a^* given L is

$$a^* = \operatorname*{argmin}_{a \in [0,A]} \pi_G(L, a)K + c(a) = \operatorname*{argmin}_{a \in [0,A]} \tilde{\pi}_G(L)\tilde{g}(a)K + c(a). \tag{8}$$

Since a^* is the maximizer of a continuous function on the compact set $[0, A]$, then it is well defined for all L. Denote by $a^*(L)$ the optimal policy function. If we plug the optimal policy function into the intervention function $\tilde{g}(.)$ we get

$$g(L) \equiv \tilde{g}(a^*(L)). \tag{9}$$

This function will be called the *fully informed intervention function*. Using this function, we can write posteriors as

$$\pi_t(L) = \tilde{\pi}_t(L)g(L) = \frac{L}{\frac{1}{q_t} - 1 + L} g(L) \tag{10}$$

The framework introduced in this section (without the addition of the intervening agent) is somehow similar to the model of horse race betting in Ali (1977), with a continuum of bettors and two horses. Nonetheless, our model is more general in a number of ways. In Ali (1977) bettors do not gain any information from the market odds, and there is no feedback from these odds to the probability of a horse winning (presumably this is the case in clean races). Ali (1977) uses that model to explain the favorite-longshot bias: the tendency for longshots to win less often than what is implied by the market odds. In contrast, our focus is on understanding the effect of feedback on the information aggregation properties of the market price, when the intervening agent gains information from the market price.

3 Market equilibrium

To characterize a REE (when it exists) it will be useful to derive the distribution of traders' posteriors from the distribution of priors. For a given value of the intervention variable a, the distribution of prior beliefs about $\omega = 1$ determines the distribution of posterior beliefs about $z = 1$. We denote the distribution of posteriors beliefs in the population of traders by $R(.)$. The following lemma relates $R(.)$ to the distribution of prior beliefs $F(.)$:

Lemma 1 *Given a value for the intervention variable a and the distribution of prior beliefs F, the distribution of posterior beliefs $R(.)$ is given by*

$$R(x) = \begin{cases} F\left(\frac{x}{[\tilde{g}(a)-x]L+x}\right) & x \in [0, \tilde{g}(a)] \\ 1 & x > \tilde{g}(a). \end{cases} \tag{11}$$

Proof: First, notice that no trader can ever have a posterior belief above $\tilde{g}(a)$, since they recognize that the probability of $z = 1$ is the product of $\tilde{\pi}_t \leq 1$ and $\tilde{g}(a)$ (see equation (5)). Next, for $x \in [0, \tilde{g}(a)]$

$$\begin{aligned} R(x) &= \mu\Big(\Big\{t : \frac{L}{\frac{1}{q_t} - 1 + L}\tilde{g}(a) \leq x\Big\}\Big) \\ &= \mu\Big(\Big\{t : q_t \leq \frac{x}{[\tilde{g}(a) - x]L + x}\Big\}\Big) = F\Big(\frac{x}{[\tilde{g}(a) - x]L + x}\Big) \end{aligned}$$

where the rearrangements of the inequalities inside μ are justified by the fact that $\frac{1}{q_t} - 1 + L \geq 0$ for $q_t \in [0, 1]$ and $[\tilde{g}(a) - x]L + x \geq 0$ when $x \in [0, \tilde{g}(a)]$. ∎

We denote by p the price of the contingent claim on $z = 1$. The equilibrium price of the contingent claim, its quantity demanded and supplied, and the action taken by agent G will be determined simultaneously by the condition that the market clears in equilibrium and that all agents (traders as well as G) behave optimally. Given risk neutrality, traders for whom the price of the security is lower than their posterior, i.e. $p < \pi_t$, invest their whole trading budget in $\frac{y_t}{p}$ units of the contingent claim, and traders for whom $p > \pi_t$ sell as many units of the contingent claim as possible ($\frac{y_t}{1-p}$ contracts).[12] When analyzing equilibria we

[12] The measure of traders for whom $\pi_t = p$ os zero for all p. Also, we restrict sales to the maximum number of shares that can be honored with the trading budget when $z = 1$.

will distinguish two situations: first, we will look at the case in which $z = 1$ is never totally prevented by G, either because this is not possible or because it is never optimal for G to do so. We will call this the no-shutdown case. And next we will analyze the situation in which $z = 1$ can and will be totally prevented by G in some circumstances. This will be called the possible shutdown case.

3.1 The no-shutdown case

We first analyze the case in which $z = 1$ cannot be (or will never optimally be) totally prevented by G. This will be the case, for example, if $D > A$, i.e. when G's available resources are not enough to completely prevent $z = 1$. Notice that in this case $g(L) > 0$ for all L. A *full communication equilibrium* (FCE) is an equilibrium in which traders and G act according to their posteriors conditional on L and markets clear. This is, a FCE is the equilibrium that would obtain if all agents observed L. If a FCE exists, we say that it is *fully revealing* if the equilibrium price is a one-to-one function of L. It is well known that if a FCE is fully revealing, then it is a (fully revealing) REE (see for example Radner (1979)). The following proposition establishes the existence of a FCE and characterizes it:

Proposition 1 *In the prediction market model with intervention, assume that $g(L) > 0$ for all L. Then there is a unique full communication equilibrium (FCE) with price function given implicitly by*

$$p^* = 1 - R(p^*) = 1 - F\Big(\frac{p^*}{[g(L) - p^*]L + p^*}\Big) \tag{12}$$

with $0 < p^* < g(L)$.

Proof: For each L, in equilibrium demand for the contingent claim must equal supply, that is

$$\frac{\bar{y}}{p^*}\int_{p^*}^{1} dR(\pi) = \frac{\bar{y}}{1-p^*}\int_{0}^{p^*} dR(\pi)$$

where $\bar{y}$ is the mean trading budget. Simple algebra shows that $p^* = 1 - R(p^*)$ is the only price that clears the market. Lemma 1 justifies the second equality in equation (12).

To show existence, we must show that for each L there is $p^*(L)$ which solves (12). For any given value of $L > 0$, $0 < g(L) < 1$. Define $f(p) \equiv 1 - F\Big(\frac{p}{[g(L)-p]L+p}\Big) - p$. Since F is assumed to be continuous, so is f. Also, $f(0) = 1$ and $f(g(L)) = -g(L) < 0$. Therefore, by the Intermediate Value Theorem there is $p^* \in (0, g(L))$ such that $f(p^*) = 0$, this is, $p^* = 1 - F\Big(\frac{p^*}{[g(L)-p^*]L+p^*}\Big)$; i.e. p^* is an equilibrium price. Furthermore, since $f(p)$ is strictly decreasing in $[0, g(L)]$, for each L the FCE price is unique. ∎

Notice that the result in Proposition 1 hinges critically on the assumption that $g(L) > 0$ for all L, i.e. that $z = 1$ cannot be totally prevented by G. This provides a best case scenario for information aggregation. (Yet, our result below show that even in this most favorable case the market may fail to aggregate information.) We will explore the consequences of relaxing this assumption later.

It is worth noticing that our model nests the case without the possibility of intervention, analyzed by Manski (2006), Wolfers and Zitzewitz (2007), Ottaviani and Sørensen (2015) and others. No intervention in our model corresponds to $A = 0$, with $g(0) = 1$. In that case the FCE is fully revealing, and therefore a REE. The following corollary summarizes these results:

Corollary 1 *If $A = 0$, then the FCE price function is given by*

$$p^* = 1 - F\Big(\frac{p^*}{[1-p^*]L + p^*}\Big). \tag{13}$$

In this case, the FCE is fully revealing and therefore it is a fully revealing REE. Furthermore, the REE price function is strictly increasing in L, i.e. the price increases with information more favorable to $z = 1$.

Proof: From (13) it is easy to see equilibrium price is increasing in L. This implies that the FCE is fully revealing, and hence a REE. ∎

Corollary 1 shows that in the no-intervention case the market acts as an efficient information aggregator. The term "rational expectations equilibrium" is strange in this setting, since all agents trivially share the same information — the signal s is observed by all traders, and G is absent from the model in the no-intervention case. The interpretation of a fully revealing REE in this situation is that an outside observer could infer the value of the likelihood ratio L by observing the market price. Notice that corollary 1 does not impose any structure on the signal (or likelihood ratio) space. This means that without intervention a fully revealing REE exists even if the support of L is continuous.[13] Furthermore, the price function in this case is always monotonically increasing in L. This is an intuitive result: if no intervention is possible, a higher value of the likelihood ration L implies a higher probability of $\omega = 1$; therefore, all traders will hold higher posteriors of $z = 1$ and the market clearing price will be higher. We will see below that the presence of intervention invalidates these results. In particular, we will see that if intervention is possible: a) a (fully or partially revealing) REE may not exist when the support of the likelihood ratio is continuous; and b) if the support of the likelihood is finite or countable, a REE will (generically) exist, but the price function associated with equilibrium will not necessarily be increasing in L. In case a) the market essentially fails as an information aggregation mechanism. In case b) the market does aggregate information, but an outside observer with no knowledge of the price function will not be able to infer if information is more or less favorable from observing price variations. Example 4 illustrates that in the intervention scenario the FCE is not necessarily revealing, i.e. case a). Example 5 illustrates that a fully revealing REE price function may not be increasing in L, i.e. case b).

Example 4: Assume that the signal space is such that $supp\,(L) = (0, \infty)$, $a \in [0, 1]$ $(A = 1)$, that $\delta \sim u[0, 1]$ $(D = 1)$, and that F is also $u[0, 1]$, this is, priors are uniformly distributed on $[0, 1]$. Further, assume that the damage when $z = 1$ occurs is $K = 2$, the direct cost of intervention is $c(a) = a^2$ and G's prior beliefs about the probability of $\omega = 1$ is $q_G = \frac{1}{2}$. Given these assumptions,

$$\tilde{g}(a) = \int_a^1 d\delta = 1 - a \tag{14}$$

Plugging (14) into G's problem we get

$$\min_{a \in [0,1]} 2\tilde{\pi}_G(L)(1 - a) + a^2 \tag{15}$$

[13] When the signal space has the same dimensionality as the price space existence of a REE is not guaranteed; see for example Allen and Jordan (1998).

which has a global minimum at $a^*(L) = \tilde{\pi}_G(L)$. Therefore,

$$g(L) = 1 - \tilde{\pi}_G(L) = 1 - \frac{L}{\frac{1}{q_G} - 1 + L} = \Big(\frac{1}{1+L}\Big) \tag{16}$$

where the last equality follows from the assumed value of q_t. Notice that since $L > 0$, $0 < g(L) < 1$. By proposition 1 the FCE price is given by

$$p^* = 1 - \frac{p^*}{[\frac{1}{1+L} - p^*]L + p^*}. \tag{17}$$

We can compute from (17) that for $L = 0.13$ and $L' = 2.53$, $p^*(L) = p^*(L') = \frac{1}{4}$. Therefore, the FCE in this example is not fully revealing. ∎

Example 5: Maintain all the assumptions of example 4 except that now *supp*$(L) = \{0.13, 1, 3\}$. Again, the FCE price is given by

$$p^* = 1 - \frac{p^*}{[\frac{1}{1+L} - p^*]L + p^*}. \tag{18}$$

We can compute from (18) that $p^*(0.13) = \frac{1}{4}$, $p^*(1) = \frac{1}{3}$ and $p^*(3) = 0.2276$. Therefore, the FCE is a fully revealing REE, but the equilibrium price first increases and then decreases in in light of information more favorable of $\omega = 1$. Therefore an outside observer without knowledge of the price function will not be able to infer *even the direction* of changes in information from observing price movements. ∎

The intuition behind example 5 is the following. On one hand, when L increases the market recognizes that $\omega = 1$ is more likely, which for a given level of response, makes the probability of $z = 1$ higher. On the other hand, the market also recognizes that in the revealing equilibrium a higher likelihood ratio triggers a higher response by G, decreasing the probability of $z = 1$. This, in turn, leads traders to revise their posterior beliefs of $z = 1$ downwards. The net effect can go either way—a higher probability of $\omega = 1$ may well cause a *decrease* in the price of the claim that is contingent on $z = 1$. In consequence, an outside observer cannot rely on price changes alone to make inferences about the direction of the change in L. The absence of a monotone response of the equilibrium price to more favorable information is one of the most important distortions introduced in the market by the intervening agent, since in this case changes in prices cannot be interpreted as changes in information.

While proposition 1 asserts the existence of a FCE, the interesting question is that of the existence an properties of a fully or partially revealing REE, that is, an equilibrium that transmits all or some information. Corollary 1 showed that such an equilibrium always exists in the no intervention case. In contrast, Proposition 2 asserts that a REE may not exist when intervention is present.

Proposition 2 *In the prediction market model with intervention assume that* $g(L) > 0$ *for all* L*. Then*

1. *If the unique FCE characterized in proposition 1 is fully revealing, then it is also the unique fully revealing REE with price function given implicitly by (12).*
2. *If the FCE of proposition 1 is not fully revealing, then there is no (fully or partially revealing) REE.*

Proof:

1. A fully revealing FCE is always a fully revealing REE with the corresponding price function (c.f. Radner (1979)).
2. We rewrite the market clearing condition (12), denoting by $p^*(L)$ the (not fully revealing) price function in the FCE.

$$p^*(L) = 1 - F\Big(\frac{p^*(L)}{[g(p^*) - p^*(L)]L + p^*(L)}\Big) \tag{19}$$

First, assume that the FCE is a REE. Since the FCE is not fully revealing, there is at least a pair (L, L') with $L < L'$ for which $p^*(L) = p^*(L')$. In a REE therefore we must have $g(p^*(L)) = g(p^*(L'))$. But then the same price cannot satisfy (19) for $g(p^*(L)) = g(p^*(L'))$ and $L < L'$.

Next, assume that there is a REE different from the FCE. This equilibrium cannot be fully revealing, or otherwise it would be a FCE. Hence, it can be at most partially revealing. This means that there is at least a pair (L, L') with $L < L'$ for which $p(L) = p(L')$, and consequently $g(p(L)) = g(p(L')) > 0$. But since trader's posteriors are strictly increasing in L for a fixed value of g, the market clearing price under L' must be higher than that under L, a contradiction.

∎

The results in proposition 2 highlight the substantial differences in the information aggregation properties of the market with and without intervention. While in the no intervention case a REE always exists and the market price fully reveals all traders' information (corollary 1), in a setting with intervention a REE exists *only in the special case in which the FCE price function one-to-one.* Example 4 illustrates a case in which this does not happen, i.e. a REE does not exist. When the FCE price function (12) implicitly defines a many-to-one function $p^*(L)$, there is no equilibrium, not even a partially revealing one. This means that the market unravels, and therefore it does not convey any of the traders' information to G, i.e. the market does not fulfill its desired function as an information aggregator.

Given the importance of monotonicity of the FCE price function to have a REE, it is interesting to establish conditions under which such function is indeed monotonic. Proposition 3 establishes sufficient conditions for the FCE to be monotonic, and therefore for a fully revealing REE to exist.

Proposition 3 *Assume that* $\text{supp}(L)$ *is an open subset of* $\mathbb{R}$ *and that* F *and* $g(.)$ *are differentiable. Then (12) defines implicitly a differentiable function* $p^*(L)$ *and for all* $L \in \text{supp}(L)$ *the sign of* $\frac{dp^*}{dL}(L)$ *is the same as the sign of* $(g'L + g - p)$*.*

Proof: From (12) define

$$Q(p, L) = -p + 1 - F\Big(\frac{p}{[g(L) - p]L + p}\Big) \tag{20}$$

Since F and g are differentiable, we can apply the implicit function theorem to get

$$\frac{dp(L)}{dL} = -\frac{\frac{\partial Q}{\partial L}}{\frac{\partial Q}{\partial p}} = \frac{F'p(g'L + g - p)}{[(g - p)L]^2 + F'gL} \tag{21}$$

By proposition 1, the denominator in (21) is positive at the equilibrium price. Also, $F' > 0$ at the equilibrium price since F' is assumed to be strictly increasing in $[0, 1]$ and by proposition 1 the equilibrium price is positive. Therefore

$$\frac{dp(L)}{dL} \gtreqless 0 \iff (g'L + g - p) \gtreqless 0 \tag{22}$$

∎

To gain some intuition about condition (22) in proposition 3, it is useful to think about the change in the posterior of the *marginal trader* when L changes. The marginal trader is the trader who is indifferent between buying or selling the asset, that is, the marginal trader is that for who the posterior is equal to the equilibrium price:[14]

$$p = \Pi_{marg}(L) = \frac{L}{\frac{1}{q_{marg}} - 1 + L} g(L), \tag{23}$$

which gives

$$\frac{1}{q_{marg}} = \frac{Lg}{p} + 1 - L. \tag{24}$$

Differentiating $\Pi_{marg}(L)$ with respect to L and plugging in (24), we can compute the change in the posterior of the marginal trader for a small change in L to be

$$\frac{\partial \Pi_{marg}(L)}{\partial L} = g'L + g - p \tag{25}$$

which is precisely condition (22). The intuition is the following: there are two effects of a higher L in the distribution of posteriors about $z = 1$. First, traders recognize that $\omega = 1$ is more likely, which makes them revise their posteriors upwards. Second, G also revises its posterior upwards, which triggers a higher intervention level. This in turn makes traders revise their posteriors downwards. The net effect may be either an increase in posteriors for some traders and a decrease for some others, depending on the current value of L and the priors q_t. However, the equilibrium price will go up (down) if and only if the posterior of the *marginal trader* goes up (down). This can be seen by noticing that the posteriors (10) are increasing in the priors q_t. Therefore, if the posterior of the marginal trader increases,

[14] Since the distribution of priors is continuous and posteriors are strictly increasing in priors, the marginal trader(s) have measure zero.

the measure of traders who want to buy the asset increases (since the posteriors of traders with higher priors are still higher than that of the marginal trader). As a consequence, the price has to increase to clear the market again. Therefore, the question of whether the price increases or decreases as a result of an increase in the likelihood ratio L is equivalent to the question of whether the posterior of the marginal trader increases or decreases due to the increase in L, which is precisely condition (22) of proposition 3.

Next we move to the possible shutdown case. In that case the occurrence of $z = 1$ can be completely prevented by G in some cases. We will see that the potential of shutdown will make results to differ even more from the no intervention case compared with the no-shutdown case. In particular, if a REE exists, it will be only partially revealing. Furthermore, we will show the possibility of completely non-revealing equilibria in which the market price contains no information received by traders.

3.2 The possible shutdown case

An important condition to ensure existence of a fully revealing FCE in Proposition 1 was that $g(L) > 0$ for all $L > 0$. That is, it was assumed that under no circumstance G would want (or could) totally prevent the occurrence of $z = 1$. In certain situations this assumption may not be realistic. For example, a security agency could shut down an airport, hence preventing any terrorist attack on that site. It is interesting to inquire if there could exist equilibria (possibly partially or non-revealing) if the no-shutdown condition is relaxed. Propositions 5 and 6 answer this question.

First, in Proposition 4 we show that if it is possible for G to totally prevent $z = 1$, and there are at least two values of L for which shutdown is optimal, the FCE cannot be fully revealing. The intuition is the following: if $z = 1$ is totally prevented from happening, then for any positive price all traders would want to take short positions in the contingent claim, so the only possible equilibrium price is zero. Since this is the case for two different values of L, the FCE cannot be fully revealing.

Proposition 4 *Assume that $A \geq D$, this is, $\tilde{g}(a) = 0$ for $a \in [D, A]$. If there is a pair*

$L \neq L'$ such that

$$a^*(L) = a^*(L') = D, \tag{26}$$

then the unique FCE is not fully revealing.

Proof: First, for L such that $g(L) > 0$, equation (12) gives the unique FCE price. Next, for L such that $a^*(L) = D$ we have that $g(L) = 0$ and from equation (5) we have $\pi_t(L, D) = \pi_t(L', D) = 0$ for all t. Since then traders' posteriors are all equal to zero, given risk neutrality, all traders will want to sell the asset at any positive price. The only price that leaves traders indifferent is $p^* = 0$. Hence $p^*(L) = 0$. Therefore, the FCE price function is given by (12) for L such that $g(L) > 0$ and $p^*(L) = 0$ for L such that $g(L) = 0$.

Next, since there is a pair (L, L') for which $a^*(L) = a^*(L') = D$ and then $g(L) = g(L') = 0$, we have that $p^*(L) = p^*(L') = 0$, i.e. the FCE is not fully revealing. ∎

Proposition 4 suggest that when shutdown is possible, there could exist a partially revealing REE with price function given by (12) for values of the likelihood ratio that do not call for shutdown and $p^*(L) = 0$ for values of L for which shutdown is optimal for G. The following proposition gives conditions for existence of such equilibrium.

Proposition 5 *Assume that $A \geq D$, this is, $\tilde{g}(a) = 0$ for $a \in [D, A]$, and that there is at least a pair $L \neq L'$ such that*

$$a^*(L) = a^*(L') = D, \tag{27}$$

Define the set $N = \{L \in \text{supp}(L) | a^*(L) = D\}$. Then

1. *If (12) defines a one-to-one function $p^*(L)$ for $L \in N^c$, there is a partially revealing REE with price function given by $p^*(L) = 0$ for $L \in N$ and by (12) for $L \in N^c$.*

2. *If (12) defines a many-to-one function $p^*(L)$ for $L \in N^c$, there is no REE.*

Proof:

1. Assume that G and all traders have the proposed price function. We need to check that the market clears and that the predicted price is confirmed for all L. First take

$L \in N$, this is, $p^*(L) = 0$. Then $a^*(L) = D$ and $\pi_t(L, D) = 0$ for all t, so the market clears with $p^* = 0$. Next, take $L \in N^c$ (notice that N^c may be empty). For such L the price function is one-to-one by hypothesis, and $g(L) > 0$. Hence, by proposition 1 the price given implicitly by (12) clears the market.

2. Since the price function is not one-to-one on N^c, there is at least a pair $L \neq L'$ with $L, L' \in N^c$ such that $p^*(L) = p^*(L')$. Then $g(p^*(L)) = g(p^*(L')) > 0$. The argument in the proof of proposition 2 confirms that there is no REE.

∎

The following example shows that proposition 5 is not vacuous.

Example 6: Modify example 4 by assuming that $\delta \in \{0, \frac{1}{2}\}$, with $Prob[\delta = \frac{1}{2}] = v$ (this is, $D = \frac{1}{2}$). Then

$$\tilde{g}(a) = \begin{cases} v & a \in [0, \frac{1}{2}) \\ 0 & a \in [\frac{1}{2}, 1] \end{cases} \tag{28}$$

Since $c(a) = a^2$ (strictly increasing in $[0, 1]$), the optimal action in a FCE is given by

$$a^*(L) = \begin{cases} 0 & \tilde{\pi}_G(L) < \frac{1}{4Kv} \\ \frac{1}{2} & \tilde{\pi}_G(L) \geq \frac{1}{4Kv} \end{cases} \tag{29}$$

Assume that $K = 2$ and $v = \frac{1}{2}$ (and remember that $\tilde{\pi}_G(L) = \frac{L}{1+L}$). The optimal action in a FCE is

$$a^*(L) = \begin{cases} 0 & L < \frac{1}{3} \\ \frac{1}{2} & L \geq \frac{1}{3} \end{cases} \tag{30}$$

The corresponding fully informed intervention function $g(L)$ is

$$g(L) = \begin{cases} \frac{1}{2} & L < \frac{1}{3} \\ 0 & L \geq \frac{1}{3} \end{cases} \tag{31}$$

Therefore the FCE price function is given by

$$p \quad = 1 - \tfrac{p}{[\frac{1}{2}-p]L+p} \quad L < \frac{1}{3} \tag{32}$$

$$p \quad = 0 \quad L \geq \frac{1}{3} \tag{33}$$

The implicit price function in (32) is monotonically increasing for $L \in [0, \frac{1}{3})$ and constant in $L \in [\frac{1}{3}, \infty)$, this is, the FCE is partially revealing. Hence, there is a partially revealing REE with price function given by (32)-(33). ∎

Notice that while the equilibrium in example 6 is only partially revealing, the equilibrium price reveals all *relevant* information to G, since G takes an optimal action for all values of L. On the other hand, an outside observer can only infer the value of L from the price if L is below the shutdown threshold, and can only infer that it is above the threshold when the event is shut down.

Next, in Proposition 6 we show that if G's prior beliefs call for shutdown (preventing $z = 1$), then there is a non-revealing REE in which the price of the asset is equal to zero for all values of L, and the event is indeed shut down. Intuitively, if the event is shut down according to G's prior beliefs alone, the only possible price is zero. In that case the equilibrium price does not reveal any information about L to G. Therefore, G acts optimally based on its prior belief.

Proposition 6 *Assume that $A \geq D$, so $\tilde{g}(a) = 0$ for $a \in [D, A]$. If*

$$\underset{a\in[0,A]}{\operatorname{argmin}}\, q_G \tilde{g}(a)K + c(a) = D, \tag{34}$$

then there is a non-revealing REE, with equilibrium price function $p^*(L) = 0$ for all L and and G's optimal action $a^* = D$.

Proof: The condition that

$$D = \underset{a\in[0,A]}{\operatorname{argmin}}\, q_G g(a)K + c(a)$$

means that when G acts following solely its prior beliefs, its optimal action involves shutting down $z = 1$. In that case, rational traders cannot assign positive probability to $z = 1$, regardless of the signal received. Indeed, since $\tilde{g}(D) = 0$, we know from equation (5) that the posterior probability of $z = 1$ is equal to zero for all traders, regardless of the signals received. For those posteriors all traders would want to sell the asset at any positive price. The only price that leaves traders indifferent is $p^* = 0$. It is trivial that this price does not reveal any information about L, as the price is a constant function of L. Therefore, G cannot update its beliefs, and therefore it acts following its prior belief q_G, as condition (34) shows.

Regarding G's optimal action, since $c(a)$ is increasing and $g(a) = 0$ for $a \in [D, A]$, then $a^* = \text{argmin}_{a \in [0,A]} q_G g(a) K + c(a) \in [D, A]$ implies that $a^* = D$. ∎

Proposition 6 tells us that in some cases high initial expectations about the occurrence of an event may cause the event to be preemptively shut down, preventing the market from aggregating information. In particular, high initial expectations may trigger intervention by themselves, and if the intervention prescribed by G's cost minimization is too strong, the only price at which the market could operate is zero. In that case, the contingent market unravels and no information is gained from it (notice that there is no trade in this non-revealing equilibria). An example of this situation would be a government agency shutting down an airport in response to high prior beliefs of a terrorist attack. In that case the price of the security would be equal to zero for any information received by traders, and no information could be gained about L from the market price.

The simultaneous nature of REE makes it somewhat difficult to understand how this result really works. In a non-revealing REE, G anticipates the market not reflecting any information, and therefore acts according to its prior beliefs q_G, choosing a strong response ($a^* = D$). Traders anticipate G's a strong intervention level, and therefore are unwilling to pay any positive price for the contingent claim. The equilibrium price and actions confirm both G's and traders' beliefs: the equilibrium price does not reveal any information, since it is a constant function of L, and trader's beliefs about G's strong action are confirmed in equilibrium.

It is worth noticing that in the case of proposition 6, there could also be a partially revealing equilibrium. Indeed, condition (34) implies that $a^*(L) = D$ for all $L \geq 1$. But it could be the case that $a^*(L) < D$ for some $L < 1$. In this case, proposition 5 gives conditions for existence of a partially revealing REE. When these two equilibria coexist, the partially revealing one is preferred from G's perspective, since shut down may be suboptimal for some values of L which could be revealed in the partially revealing equilibrium. Therefore, when the conditions of proposition 6 are satisfied, G may find it optimal not to shutdown preemptively. The following example illustrates this point.

Example 7: Assume that the market is as in example 6. Since $q_t = \frac{1}{2}$, there is an equilibrium in which $p^* = 0$ for all L, and $a^* = \frac{1}{2}$ for $p = 0$. To see this, notice that when using its prior, G's optimal action is

$$a^* = \begin{cases} 0 & q_G < \frac{1}{4} \\ \frac{1}{2} & q_G \geq \frac{1}{4} \end{cases} \tag{35}$$

Therefore the optimal action when L is not revealed is $a^* = \frac{1}{2}$, this is, $g(a^*) = 0$. Given this all traders hold posteriors equal to zero, and therefore the only price which clears the market is $p^* = 0$. This is a completely non-revealing equilibrium. ∎

Comparing examples 6 and 7 we see that in this case there are two equilibria, one that is partially revealing and one that is completely non-revealing. The former is preferred by G. This is because G's action is the same in both equilibria for $L \geq \frac{1}{3}$ but in the partially revealing equilibrium G chooses the optimal action conditional on information also for $L < \frac{1}{3}$, while in the non-revealing equilibrium it chooses a suboptimal action (shutdown) for $L < \frac{1}{3}$.

4 Conclusion

The literature on contingent markets has overlooked the potential feedback effects that might arise when the market price is used as a guide in decision making. The current article is, to the best of our knowledge, the first attempt to incorporate such feedback into a theoretical model of a market trading in Arrow-Debreu contingent securities.

We have shown that the presence of an intervening agent in a contingent market opens up a new set of possibilities for the information aggregation properties of the market. In particular, we showed that while a fully revealing REE could exists under some special conditions, we also show that there are situations in which (i) a REE does not exist; (ii) partially revealing equilibria exist; (*iii*) a non-revealing equilibrium exists in which the underlying event is shut down and the price of the security is zero regardless of the information held by traders. The usefulness of the contingent market as an aid for decision making is severely diminished when it is impossible to recover information from the market price.

Furthermore, we show that even when a fully revealing REE exists, the equilibrium price may respond *negatively* to information more favorable to the occurrence of the event in which the market trades on. This novel result arises because the effects of more favorable information are twofold: First, given an intervention level, more favorable information induces higher posteriors for traders; this is the familiar effect of more favorable information, which is the only one at play in the no-intervention case. Second, more favorable information induces a higher level of response by the intervening agent, thus inducing lower posterior beliefs for traders. When this second effect dominates, the equilibrium price reacts negatively to information ex-ante more favorable to the occurrence of the underlying event.

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